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Coefficients of Polynomials of Restricted Growth on the Real Line

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Let $\phi\colon (-\infty, \infty) \to (0, \infty)$ be a given continuous even function and let m be a positive integer. We show that, with some additional restrictions on ϕ, there exist decreasing sequences $x_1, ..., x_m$ and $y_1, ..., y_{m-1}$ of symmetrically located points on $(-\infty, \infty)$ and corresponding polynomials P and Q of degrees $m-1$ and m, respectively, satisfying

$$|P(x)| \leqslant \phi(x)^m, \qquad |Q(x)| \leqslant \phi(x)^m, \qquad -\infty < x < \infty,$$

where equality holds with alternating signs at the corresponding sequence of points (and also at $\pm\infty$ for Q). Moreover, for any polynomial p of degree at most m,

(a) if $|p(x_j)| \leqslant \phi(x_j)^m$ for $j=1, ..., m$, then $|p^{(k)}(0)| \leqslant |P^{(k)}(0)|$ whenever k and m have opposite parity and $0 \leqslant k < m$;

(b) if $|p(y_j)| \leqslant \phi(y_j)^m$ for $j=1, ..., m-1$ and if $\limsup_{y\to\infty} |p(y)|/\phi(y)^m \leqslant 1$, then $|p^{(k)}(0)| \leqslant |Q^{(k)}(0)|$ whenever k and m have the same parity and $0 \leqslant k \leqslant m$.

We give two computational methods for determining these sequences of points and thus P and Q. © 1998 Academic Press

1. INTRODUCTION

Let ϕ be a continuous even function on the real line with positive values. We consider the class of all real polynomials p of degree at most m satisfying $|p(x)| \leqslant \phi(x)^m$ for all real numbers x. Our object is to determine the maximum possible absolute value of each coefficient for polynomials in this class. We show that with some restrictions on ϕ, for each positive integer m there exist extremal polynomials P and Q in the class with the property that their nonzero coefficients have the largest absolute value of any polynomial p in the class. Thus we obtain the best bounds on all the coefficients of p since the nonzero coefficients of P and Q are the coefficients of the alternate decreasing powers of x beginning with x^{m-1} and x^m, respectively.

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We show that the same coefficient bounds continue to hold for polynomials p which are required to satisfy the inequality $|p(x)| \leqslant \phi(x)^m$ at only certain finite sequences of points where P or Q oscillates between the given upper bound and its negative. We call these sequences optimal sequences of ϕ-oscillation points and (ϕ, ∞)-oscillation points, respectively. One way of interpreting our definition of the latter is that the points ∞ and $-\infty$ are implicitly included in the oscillation points.

A key result is that under mild restrictions on ϕ, for each m there exist optimal sequences of ϕ- and (ϕ, ∞)-oscillation points. The associated extremal polynomials P and Q are obtained by Lagrange interpolation at the points of oscillation. Moreover, these polynomials depend only on ϕ and m, and inherit some of the symmetry properties of ϕ.

We give two methods to compute the mentioned optimal sequences. The first is to minimize associated functions f and g of several variables which resemble divided differences. The ordered coordinates of points where f and g assume a minimum are the desired optimal sequences defining P and Q, respectively. This method has the advantage that it applies as long as an optimal sequence of the type desired exists.

A second method is to use the constructions of optimal sequences given in the proof of their existence. Here, the optimal sequences are obtained from the alternation points for the difference between a continuous function and its best uniform approximation by a Haar system of functions. These points can be computed by the Remez algorithm.

As an example, we compute the extremal polynomials P and Q for the case where $\phi(x) = 1 + |x|$ and $1 \leqslant m \leqslant 20$. Applications of this solution to inequalities for polynomials on normed linear spaces have been given by the author in [6]. In particular, one obtains the best constants in a Markov inequality for homogeneous polynomials on normed linear spaces.

In some cases, as for instance when $\phi(x) = (1 + x^2)^{1/2}$, it is possible to obtain explicit formulas for P and Q directly. In this case our results imply some classical inequalities of Bernstein. Moreover, we can obtain the minimum values of the associated functions f and g from the coefficients of P and Q, respectively.

Our main tools are only the Chebyshev alternation theorem and the Lagrange interpolation formula applied in the manner of Rogosinski [10]. See [8, Chapt. 6] for a summary of known inequalities of this type.

2. THE EXTREMAL POLYNOMIALS

Throughout, ϕ denotes a positive continuous function on $(-\infty, \infty)$ satisfying $\phi(-x) = \phi(x)$ for all real x. In the various cases we consider, we also require several of the restrictions on ϕ given below:

(i) $\phi(x) \geqslant \phi(0)$,

(ii) $\lim_{x \to \infty} \phi(x)/x = A, \qquad A \neq 0$,

(iii) $\phi(x) \geqslant A\,|x|$,

(iv) $|x|\,\phi(1/x) = \phi(x), \qquad x \neq 0$.

Note that all of these conditions hold when both (i) and (iv) hold. For example, the function $\phi(x) = (1 + |x|^p)^{1/p}$ satisfies all of the conditions (i)–(iv) with $A = 1$ when $0 < p < \infty$.

The parity of a number or polynomial refers to whether it is even or odd. Let m be a positive integer. Given a decreasing sequence X of numbers $x_1, ..., x_m$, set

$$\omega_j(x, X) = \prod_{i \neq j} (x - x_i), \qquad \omega(x, X) = \prod_{i=1}^{m} (x - x_i).$$

When $m = 1$, take $\omega_j(x, X) \equiv 1$. Define (for the case of opposite parity)

$$f(x_1, ..., x_m) = \sum_{j=1}^{m} \frac{\phi(x_j)^m}{|\omega_j(x_j, X)|}, \tag{1}$$

$$P(x) = \sum_{j=1}^{m} (-1)^{j-1}\, \phi(x_j)^m \frac{\omega_j(x, X)}{\omega_j(x_j, X)}. \tag{2}$$

Let Y be a decreasing sequence of numbers $y_1, ..., y_{m-1}$, where $m > 1$, and suppose ϕ satisfies (ii). Define (for the case of the same parity)

$$g(y_1, ..., y_{m-1}) = \frac{A^m}{2} \left(\sum_{j=1}^{m-1} y_j^2 \right) + \sum_{j=1}^{m-1} \frac{\phi(y_j)^m}{|\omega_j(y_j, Y)|}, \tag{3}$$

$$Q(x) = A^m x\, \omega(x, Y) - \sum_{j=1}^{m-1} (-1)^{j-1}\, \phi(y_j)^m \frac{\omega_j(x, Y)}{\omega_j(y_j, Y)}. \tag{4}$$

It follows from formulas (12) and (13) given below that $f(x_1, ..., x_m)$ is the coefficient of x^{m-1} in $P(x)$ and that $g(y_1, ..., y_{m-1})$ is the negative of the coefficient of x^{m-2} in $Q(x)$ when $\sum_1^{m-1} y_j = 0$ and $m > 1$. The polynomial P above is characterized by the condition that P has degree at most $m - 1$ and

$$P(x_j) = (-1)^{j-1}\, \phi(x_j)^m, \qquad j = 1, ..., m. \tag{5}$$

The polynomial Q above with $\sum_1^{m-1} y_j = 0$ is characterized by the condition that Q is a polynomial that has no term of degree $m - 1$ and

$$Q(y_j) = (-1)^j \phi(y_j)^m, \qquad j = 1, ..., m-1, \tag{6}$$

$$\lim_{x \to \infty} \frac{Q(x)}{\phi(x)^m} = 1. \tag{7}$$

To see the relation between P and Q, add two additional points to the sequence $y_1, ..., y_{m-1}$ by defining $y_0 = t$ and $y_m = -t$, where $t > y_1$ and $-t < y_{m-1}$. Let P_t be the polynomial of degree at most m satisfying $P_t(y_j) = (-1)^j \phi(y_j)^m$ for $j = 0, ..., m$. When the first and last terms in the Lagrange interpolation formula for the new sequence are combined, it follows from (ii) that if $\sum_1^{m-1} y_j = 0$, then

$$\lim_{t \to \infty} P_t(x) = Q(x), \qquad -\infty < x < \infty, \tag{8}$$

where Q is given by (4). (The requirement that $\sum_1^{m-1} y_j = 0$ may be omitted but this complicates somewhat the expressions for Q and g. In most of our discussion, the sequence Y will be symmetric with respect to the origin.)

3. OPTIMAL OSCILLATION POINTS

DEFINITION. Let m be a positive integer. We call a decreasing sequence of numbers $x_1, ..., x_m$ an *optimal sequence of ϕ-oscillation points* if the corresponding interpolating polynomial P given by (2) satisfies $|P(x)| \leqslant \phi(x)^m$ for all real x. We call a decreasing sequence of numbers $y_1, ..., y_{m-1}$ an *optimal sequence of (ϕ, ∞)-oscillation points* if $\sum_1^{m-1} y_j = 0$ and the corresponding interpolating polynomial Q given by (4) satisfies $|Q(x)| \leqslant \phi(x)^m$ for all real x.

We shall see later (Proposition 4 below) that for a given m, the polynomials P and Q in the preceding definition are independent of the particular sequence of oscillation points used to define them. Thus, for example, if the equation $|P(x)| = \phi(x)^m$ has only m solutions, no other optimal sequence of ϕ-oscillation points with m points exists. The analogous result holds also for (ϕ, ∞)-oscillation points.

PROPOSITION 1. *Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ be a polynomial satisfying $|p(x)| \leqslant \phi(x)^m$ for all real x, and let f and g be given by* (1) *and* (3). *Then*

$$|a_{m-1}| \leqslant f(x_1, ..., x_m) \tag{9}$$

whenever $x_1, ..., x_m$ *are distinct real numbers. If* $m > 1$ *and* (ii) *holds, then*

$$|a_{m-2}| \leqslant \frac{A^m}{2} \left(\sum_{j=1}^{m-1} y_j \right)^2 + g(y_1, ..., y_{m-1}) \tag{10}$$

whenever $y_1, ..., y_{m-1}$ *are distinct real numbers. Equality holds in* (9) *if* $x_1, ..., x_m$ *is an optimal sequence of* ϕ*-oscillation points and* $p = P$. *Equality holds in* (10) *if* $y_1, ..., y_{m-1}$ *is an optimal sequence of* (ϕ, ∞)*-oscillation points and* $p = Q$.

The following corollary follows immediately.

COROLLARY 2. *Given a positive integer m, the function f attains its minimum at any optimal sequence of* ϕ*-oscillation points with m points. The function g attains its minimum over points with* $\sum_1^{m-1} y_j = 0$ *at any optimal sequence of* (ϕ, ∞)*-oscillation points with* $m-1$ *points.*

Proof. Since f and g are symmetric functions, we may reorder the x_j's and y_j's into decreasing sequences X and Y. To prove (9), note that by replacing $p(x)$ by $[p(x) - (-1)^m p(-x)]/2$, we may suppose that the degree of p is at most $m-1$. Then by the Lagrange interpolation formula,

$$p(x) = \sum_{j=1}^{m} p(x_j) \frac{\omega_j(x, X)}{\omega_j(x_j, X)}. \tag{11}$$

Equating the coefficients of x^{m-1} on each side, we have that

$$a_{m-1} = \sum_{j=1}^{m} \frac{p(x_j)}{\omega_j(x_j, X)}, \tag{12}$$

and hence $|a_{m-1}| \leqslant f(x_1, ..., x_m)$.

Now let $s_1 = \sum_1^{m-1} y_j$ and $s_2 = \sum_{i<j} y_i y_j$, and observe that

$$(x + s_1)\, \omega(x, Y) = x^m + (s_2 - s_1^2)\, x^{m-2} + \cdots.$$

To prove (10), note that by replacing $p(x)$ by $[p(x) + (-1)^m p(-x)]/2$, we may suppose that $a_{m-1} = 0$. Let

$$r(x) = p(x) - a_m(x + s_1)\, \omega(x, Y).$$

Then r is a polynomial of degree at most $m-2$. Since r satisfies $r(y_j) = p(y_j)$ for $j = 1, ..., m-1$, by the Lagrange interpolation formula for r,

$$p(x) = a_m(x + s_1)\, \omega(x, Y) + \sum_{j=1}^{m-1} p(y_j) \frac{\omega_j(x, Y)}{\omega_j(y_j, Y)}.$$

Collecting the coefficients of x^{m-2} on both sides and using $s_1^2 = 2s_2 + \sum_1^{m-1} y_j^2$, we obtain

$$a_{m-2} = -\frac{a_m}{2}\left(s_1^2 + \sum_{j=1}^{m-1} y_j^2\right) + \sum_{j=1}^{m-1} \frac{p(y_j)}{\omega_j(y_j, Y)}. \tag{13}$$

Therefore, $|a_{m-2}| \leqslant A^m s_1^2/2 + g(y_1, ..., y_{m-1})$ since $|a_m| \leqslant A^m$ by (ii).

To prove the assertions about equality, let X and Y be the given optimal sequences and let P and Q be the corresponding interpolating polynomials. Let A_{m-1} be the coefficient of x^{m-1} in $P(x)$ and let B_{m-2} be the coefficient of x^{m-2} in $Q(x)$. Since the degree of P is at most $m-1$ and the sign of $\omega_j(x_j, X)$ is $(-1)^{j-1}$, it follows from (12) that $A_{m-1} = f(x_1, ..., x_m)$. In the same way, it follows from (13) that $B_{m-2} = -g(y_1, ..., y_{m-1})$. ∎

The next proposition shows that one can compute optimal sequences of ϕ- or (ϕ, ∞)-oscillation points when they exist by finding a point at which the corresponding function is minimum.

Proposition 3. *If there exists an optimal sequence of ϕ-oscillation points with m points and if f assumes its minimum at a point $(x_1, ..., x_m)$, then the coordinates of this point arranged in decreasing order are an optimal sequence of ϕ-oscillation points. If there exists an optimal sequence of (ϕ, ∞)-oscillation points with $m-1$ points and if g assumes its minimum over points whose coordinates sum to 0 at a point $(y_1, ... y_{m-1})$, then the coordinates of this point arranged in decreasing order are an optimal sequence of (ϕ, ∞)-oscillation points.*

Proposition 4. *All optimal sequences of ϕ-oscillation points with the same number of points have the same interpolating polynomial. All optimal sequences of (ϕ, ∞)-oscillation points with the same number of points have the same interpolating polynomial.*

Proof of Propositions 3 *and* 4. Let P be the interpolating polynomial corresponding to a given optimal sequence of ϕ-oscillation points with m points and let A_{m-1} be the coefficient of x^{m-1} in $P(x)$. If f assumes its minimum at a point $(x_1, ..., x_m)$, then $A_{m-1} = f(x_1, ..., x_m)$ by Proposition 1. Reorder the x_j's into a decreasing sequence X. Then by (12),

$$A_{m-1} = \sum_{j=1}^{m} \frac{P(x_j)}{\omega_j(x_j, X)} = \sum_{j=1}^{m} \frac{(-1)^{j-1} P(x_j)}{|\omega_j(x_j, X)|}$$
$$\leqslant \sum_{j=1}^{m} \frac{\phi(x_j)^m}{|\omega_j(x_j, X)|} = f(x_1, ..., x_m),$$

and hence $(-1)^{j-1} P(x_j) = \phi(x_j)^m$ for $j = 1, ..., m$. Thus X is an optimal sequence of ϕ-oscillation points and P is also the interpolating polynomial corresponding to X. This proves the first part of Proposition 3.

To prove the first part of Proposition 4, suppose now that X is another optimal sequence of ϕ-oscillation points $x_1, ..., x_m$. By Corollary 2, the function f assumes its minimum at $(x_1, ..., x_m)$. Hence by what we have shown, P is also the interpolating polynomial corresponding to X, as required.

A similar argument based on (13) establishes the case of (ϕ, ∞)-oscillation points. ∎

It is an interesting fact that if $x_1, ..., x_m$ is an optimal sequence of ϕ-oscillation points, then any polynomial p of degree at most $m-1$ satisfies the inequality $|p(x)| \leqslant \phi(x)^m$ for all real x when it satisfies this inequality for all x with $x_m < x < x_1$. (Compare with Lemmas 9 and 10 below.) To prove this, let P be the corresponding interpolating polynomial and note that $|P(x)| \leqslant \phi(x)^m$ for all real x by definition. By hypothesis, $|p(x_j)| \leqslant \phi(x_j)^m$ for $j = 1, ..., m$, and hence $|p(x)| \leqslant |P(x)|$ for all real x outside of (x_m, x_1) by an observation of Rogosinski [10, Theorem I]. Therefore, the required inequality holds in all cases.

4. MAIN RESULTS

THEOREM 5 (Opposite Parity). *Let m be a positive integer.*

(1) *Suppose ϕ satisfies* (ii) *and, when m is odd, suppose ϕ satisfies* (i) *also. Then there exists an optimal sequence of ϕ-oscillation points $x_1, ..., x_m$ which are symmetric with respect to the origin.*

(2) *Suppose there exists an optimal sequence of ϕ-oscillation points $x_1, ..., x_m$ which are symmetric with respect to the origin and let P be the corresponding interpolating polynomial. If p is a polynomial of degree at most m which satisfies*

$$|p(x_j)| \leqslant \phi(x_j)^m, \qquad j = 1, ..., m, \tag{14}$$

then

$$|p^{(k)}(0)| \leqslant |P^{(k)}(0)| \tag{15}$$

for all k with parity opposite to m and $0 \leqslant k < m$. Moreover, if equality holds in (15) *for some $k > 0$, then $[p(x) + (-1)^{m-1} p(-x)]/2 = \varepsilon P(x)$ for all real x, where $\varepsilon = \pm 1$.*

THEOREM 6 (Same Parity). *Let* $m > 1$.

(1) *If* m *is odd, suppose* ϕ *satisfies* (ii) *and* (iii), *and if* m *is even, suppose* ϕ *satisfies* (i) *and* (iv). *Then there exists an optimal sequence of* (ϕ, ∞)*-oscillation points* $y_1, ..., y_{m-1}$ *which are symmetric with respect to the origin.*

(2) *Suppose there exists an optimal sequence of* (ϕ, ∞)*-oscillation points* $y_1, ..., y_{m-1}$ *which are symmetric with respect to the origin and let* Q *be the corresponding interpolating polynomial. If* p *is a polynomial of degree at most* m *which satisfies*

$$|p(y_j)| \leqslant \phi(y_j)^m, \qquad j = 1, ..., m-1 \tag{16}$$

and

$$\limsup_{y \to \infty} \frac{|p(y)|}{\phi(y)^m} \leqslant 1, \tag{17}$$

then

$$|p^{(k)}(0)| \leqslant |Q^{(k)}(0)| \tag{18}$$

for all k *with the same parity as* m *and* $0 \leqslant k \leqslant m$. *Moreover, if equality holds in* (18) *and* $0 < k < m$, *then* $[p(x) + (-1)^m p(-x)]/2 = \varepsilon Q(x)$ *for all real* x, *where* $\varepsilon = \pm 1$.

COROLLARY 7. *Suppose* ϕ *satisfies* (i) *and* (iv). *If* p *is a polynomial of degree at most* m *such that* $|p(x)| \leqslant \phi(x)^m$ *for* $-\infty < x < \infty$, *then* $|p^{(k)}(0)| \leqslant |P^{(k)}(0)|$ *for* k *with parity opposite to* m *and* $|p^{(k)}(0)| \leqslant |Q^{(k)}(0)|$ *for* k *with the same parity as* m, *where* $1 \leqslant k \leqslant m$.

PROPOSITION 8. *Under the conditions of part* (1) *of Theorems* 5 *and* 6, *the corresponding interpolating polynomials satisfy the identities*

$$P(-x) = (-1)^{m-1} P(x), \tag{19}$$

$$Q(-x) = (-1)^m Q(x) \tag{20}$$

for all real x. *Suppose also that* ϕ *satisfies* (iv). *If* m *is even, then*

$$P(x) = (-1)^{(m-2)/2} x^m P\left(\frac{1}{x}\right), \tag{21}$$

$$Q(x) = (-1)^{m/2} x^m Q\left(\frac{1}{x}\right) \tag{22}$$

for all real $x \neq 0$, *and if m is odd, then*

$$Q(x) = (-1)^{(m-1)/2} x^m P\left(\frac{1}{x}\right) \tag{23}$$

for all real $x \neq 0$.

EXAMPLE 1. Let $\phi(x) = (1+x^2)^{1/2}$. Note that ϕ satisfies (i)–(iv) with $A = 1$. Define

$$P(x) = \operatorname{Im}(x+i)^m, \qquad Q(x) = \operatorname{Re}(x+i)^m.$$

Clearly, $|P(x)| \leqslant \phi(x)^m$ and $|Q(x)| \leqslant \phi(x)^m$ for all real x, and Q satisfies (7). We first show that P and Q are the interpolating polynomials defined in (2) and (4). Given a real number x, write $x = \cot\theta$, where $0 \leqslant \theta \leqslant \pi$. Then $\phi(x) = \csc\theta$ and

$$x + i = \frac{\cos\theta + i\sin\theta}{\sin\theta} = e^{i\theta}\csc\theta,$$

so $(x+i)^m = e^{im\theta}\phi(x)^m$. Then

$$P(x) = \sin(m\theta)\,\phi(x)^m, \qquad Q(x) = \cos(m\theta)\,\phi(x)^m.$$

Solving the equations $\sin m\theta = \pm 1$ and $\cos m\theta = \pm 1$, we see that (5) and (6) hold with

$$x_j = \cot\left(\frac{2j-1}{2m}\pi\right), \qquad j = 1, ..., m, \tag{24}$$

$$y_j = \cot\left(\frac{j}{m}\pi\right), \qquad j = 1, ..., m-1. \tag{25}$$

Clearly, (24) and (25) are the only solutions of the equations $|P(x)| = \phi(x)^m$ and $|Q(x)| = \phi(x)^m$, respectively. Hence $x_1, ..., x_m$ is the unique optimal sequence of ϕ-oscillation points with m points and $y_1, ..., y_{m-1}$ is the unique optimal sequence of (ϕ, ∞)-oscillation points with $m-1$ points. As expected, P and Q satisfy the identities of Proposition 8.

Let $p(x) = a_0 + a_1 x + \cdots + a_m x^m$. By part (2) of Theorem 5, if $|p(x_j)| \leqslant (1+x_j^2)^{m/2}$ for $j = 1, ..., m$, then $|a_k| \leqslant \binom{m}{k}$ for k with parity opposite to m. By part (2) of Theorem 6, if $|p(y_j)| \leqslant (1+y_j^2)^{m/2}$ for $j = 1, ..., m-1$ and if $|a_m| \leqslant 1$, then $|a_k| \leqslant \binom{m}{k}$ for k with the same parity as m. (Compare [1, p. 56].)

It follows from Proposition 1 that

$$\min_X \sum_{j=1}^{m} \frac{(1+x_j^2)^{m/2}}{|\omega_j(x_j, X)|} = m,$$

where X varies over all decreasing sequences of points $x_1, ..., x_m$ and, when $m > 1$,

$$\min_Y \left\{ \frac{1}{2} \left[\left(\sum_{j=1}^{m-1} y_j \right)^2 + \sum_{j=1}^{m-1} y_j^2 \right] + \sum_{j=1}^{m-1} \frac{(1+y_j^2)^{m/2}}{|\omega_j(y_j, Y)|} \right\} = \frac{m(m-1)}{2},$$

where Y varies over all decreasing sequences of points $y_1, ..., y_{m-1}$. By Proposition 3, these minima are attained when and only when (24) and (25) hold, respectively. (Compare [11] and see [5] for an application to trigonometric polynomials.)

EXAMPLE 2. Let $\phi(x) = \max\{1, |T_m(x)|^{1/m}\}$, where T_m is the classical Chebyshev polynomial of degree m and $m > 1$. Then $\phi(-x) = \phi(x)$ for all real x, ϕ is non-decreasing on $(-\infty, \infty)$, and (ii) holds with $A^m = 2^{m-1}$. Define $P(x) = T_{m-1}(x)$ and $Q(x) = T_m(x)$, and let

$$x_j = \cos\left(\frac{(j-1)\pi}{m-1}\right), \qquad j = 1, ..., m, \tag{26}$$

$$y_j = \cos\left(\frac{j\pi}{m}\right), \qquad j = 1, ..., m-1. \tag{27}$$

Then, as in Example 1, $x_1, ..., x_m$ is an optimal sequence of ϕ-oscillation points and $y_1, ..., y_{m-1}$ is an optimal sequence of (ϕ, ∞)-oscillation points. Moreover, these are the only such sequences in the interval $[-1, 1]$ with the same number of points.

Let p be a polynomial of degree at most m and let a_m be the coefficient of x^m in $p(x)$. By part (2) of Theorem 5, if $|p(x_j)| \leqslant 1$ for $j = 1, ..., m$, then $|p^{(k)}(0)| \leqslant |T_{m-1}^{(k)}(0)|$ for all k with parity opposite to m. By part (2) of Theorem 6, if $|p(y_j)| \leqslant 1$ for $j = 1, ..., m-1$ and if $|a_m| \leqslant 2^{m-1}$, then $|p^{(k)}(0)| \leqslant |T_m^{(k)}(0)|$ for all k with the same parity as m. (Compare with a classical result of Markov given in [9, pp. 53–56].)

It follows from Proposition 1 that

$$\min_X \sum_{j=1}^{m} \frac{1}{|\omega_j(x_j, X)|} = 2^{m-2},$$

where X varies over all decreasing sequences of points $x_1, ..., x_m$ in $[-1, 1]$. (Compare [3, Lemma I].) Similarly, if $m > 1$,

$$\min_Y \left\{ 2^{m-2} \left[\left(\sum_{j=1}^{m-1} y_j \right)^2 + \sum_{j=1}^{m-1} y_j^2 \right] + \sum_{j=1}^{m-1} \frac{1}{|\omega_j(y_j, Y)|} \right\} = m2^{m-3},$$

where Y varies over all decreasing sequences of points $y_1, ..., y_{m-1}$ in $[-1, 1]$. These minima are attained when and only when (26) and (27) hold, respectively.

Note that it follows from Proposition 3 that under the conditions of part (1) of Theorems 5 and 6, one can compute optimal sequences of oscillation points and the corresponding P and Q by minimizing f and g over points whose coordinates are symmetric with respect to the origin. (The symmetry reduces the number of variables in the minimization by approximately one

TABLE I

m	Extremal polynomials P for $\phi(x) = 1 + \lvert x \rvert$
1	1.0
2	$4.0x$
3	$6.9768501x^2 - 1.0$
4	$10.392305x^3 - 10.392305x$
5	$13.889998x^4 - 36.900361x^2 + 1.0$
6	$17.614678x^5 - 99.229356x^3 + 17.614678x$
7	$21.411366x^6 - 211.77494x^4 + 99.404353x^2 - 1.0$
8	$25.358580x^7 - 403.20602x^5 + 403.20602x^3 - 25.358580x$
9	$29.365943x^8 - 692.17341x^6 + 1221.0722x^4 - 201.25335x^2 + 1.0$
10	$33.485122x^9 - 1116.2572x^7 + 3189.5441x^5 - 1116.2572x^3 + 33.485122x$
11	$37.655346x^{10} - 1697.4134x^8 + 7235.2644x^6 - 4462.2666x^4 + 347.75003x^2 - 1.0$
12	$41.914185x^{11} - 2481.3769x^9 + 15067.717x^7 - 15067.717x^5 + 2481.3769x^3 - 41.914185x$
13	$46.217198x^{12} - 3492.7183x^{10} + 28857.986x^8 - 43102.021x^6 + 12477.487x^4$ $- 543.28842x^2 + 1.0$
14	$50.593432x^{13} - 4784.7605x^{11} + 52260.643x^9 - 111436.95x^7 + 52260.643x^5 - 4784.7605x^3$ $+ 50.593432x$
15	$55.008544x^{14} - 6384.1591x^{12} + 89525.390x^{10} - 260260.29x^8 + 182227.81x^6$ $- 29245.167x^4 + 791.63654x^2 - 1.0$
16	$59.485946x^{15} - 8351.3837x^{13} + 147415.59x^{11} - 567901.37x^9 + 567901.37x^7$ $- 147415.59x^5 + 8351.3837x^3 - 59.485946x$
17	$63.998042x^{16} - 10714.754x^{14} + 233318.56x^{12} - 1156556.0x^{10} + 1577544.8x^8$ $- 610763.02x^6 + 60480.150x^4 - 1096.1024x^2 + 1.0$
18	$68.564278x^{17} - 13541.531x^{15} + 358568.85x^{13} - 2242807.1x^{11} + 4057566.4x^9$ $- 2242807.1x^7 + 358568.85x^5 - 13541.531x^3 + 68.564278x$
19	$73.161829x^{18} - 16861.354x^{16} + 534972.61x^{14} - 4137163.6x^{12} + 9639233.7x^{10}$ $- 7266132.8x^8 + 1730436.5x^6 - 113920.08x^4 + 1459.6402x^2 - 1.0$
20	$77.807218x^{19} - 20747.984x^{17} + 780200.70x^{15} - 7354205.0x^{13} + 21645133.0x^{11}$ $- 21645133.0x^9 + 7354205.0x^7 - 780200.70x^5 + 20747.984x^3 - 77.807218x$

half.) This method was used in the next example. Alternately, one can use the Remez algorithm [2, p. 78] and the proofs (below) of Theorems 5 and 6, which construct the desired optimal sequences from alternation points for certain problems of best uniform approximation in a Haar space.

EXAMPLE 3. Let $\phi(x) = 1 + |x|$. Clearly ϕ satisfies (i)–(iv) with $A = 1$. The interpolating polynomials are given in Tables I and II, and it is clear that the symmetry properties expected from Proposition 8 hold. Good first approximations to optimal sequences of ϕ- and (ϕ, ∞)-oscillation points are the symmetric sequences whose positive terms are obtained by raising the positive terms of (24) and (25) to the 3/2 power. Optimal sequences were determined to at least 16 digit accuracy using the multivariate minimization subroutine LBFGS of [7] and the HP-UX Fortran 77 compiler (with precision doubling enabled) for the Hewlett–Packard HP 9000 series 700 computer. (The gradients for f and g were coded by hand without

TABLE II

m	Extremal polynomials Q for $\phi(x) = 1 + \|x\|$
2	$x^2 - 1.0$
3	$x^3 - 6.9768501x$
4	$x^4 - 18.0x^2 + 1.0$
5	$x^5 - 36.900361x^3 + 13.889998x$
6	$x^6 - 63.275280x^4 + 63.275280x^2 - 1.0$
7	$x^7 - 99.404353x^5 + 211.77494x^3 - 21.411366x$
8	$x^8 - 144.73923x^6 + 543.47846x^4 - 144.73923x^2 + 1.0$
9	$x^9 - 201.25335x^7 + 1221.0722x^5 - 692.17341x^3 + 29.365943x$
10	$x^{10} - 268.33173x^8 + 2423.4878x^6 - 2423.4878x^4 + 268.33173x^2 - 1.0$
11	$x^{11} - 347.75003x^9 + 4462.2666x^7 - 7235.2644x^5 + 1697.4134x^3 - 37.655346x$
12	$x^{12} - 438.85701x^{10} + 7644.1709x^8 - 18508.628x^6 + 7644.1709x^4 - 438.85701x^2 + 1.0$
13	$x^{13} - 543.28842x^{11} + 12477.487x^9 - 43102.021x^7 + 28857.986x^5 - 3492.7183x^3$ $+ 46.217198x$
14	$x^{14} - 660.37138x^{12} + 19427.245x^{10} - 91432.163x^8 + 91432.163x^6 - 19427.245x^4$ $+ 660.37138x^2 - 1.0$
15	$x^{15} - 791.63654x^{13} + 29245.167x^{11} - 182227.81x^9 + 260260.29x^7 - 89525.390x^5$ $+ 6384.1591x^3 - 55.008544x$
16	$x^{16} - 936.39747x^{14} + 42582.592x^{12} - 341124.71x^{10} + 664491.03x^8 - 341124.71x^6$ $+ 42582.592x^4 - 936.39747x^2 + 1.0$
17	$x^{17} - 1096.1024x^{15} + 60480.150x^{13} - 610763.02x^{11} + 1577544.8x^9 - 1156556.0x^7$ $+ 233318.56x^5 - 10714.754x^3 + 63.998042x$
18	$x^{18} - 1270.0561x^{16} + 83801.360x^{14} - 1045371.5x^{12} + 3476297.1x^{10} - 3476297.1x^8$ $+ 1045371.5x^6 - 83801.360x^4 + 1270.0561x^2 - 1.0$
19	$x^{19} - 1459.6402x^{17} + 113920.08x^{15} - 1730436.5x^{13} + 7266132.8x^{11} - 9639233.7x^9$ $+ 4137163.6x^7 - 534972.61x^5 + 16861.354x^3 - 73.161829x$
20	$x^{20} - 1664.1539x^{18} + 151935.92x^{16} - 2768847.3x^{14} + 14383919.0x^{12} - 24579265.0x^{10}$ $+ 14383919.0x^8 - 2768847.3x^6 + 151935.92x^4 - 1664.1539x^2 + 1.0$

much difficulty.) The polynomials P and Q were obtained from (2) and (4) with MAPLE V set to 35 digit accuracy and rounded to 8 digits for printing.

By the results of [6], the best constants $c_{m,k}$ in a Bernstein–Markov inequality (as well as a number of other inequalities) for normed linear spaces are given by $c_{m,k} = |P^{(k)}(0)|$ for k with parity opposite to m and $c_{m,k} = |Q^{(k)}(0)|$ for k with the same parity as m, where $0 \leqslant k \leqslant m$. (Of course, both P and Q depend on m.)

5. OPPOSITE PARITY PROOF

LEMMA 9. *Let m be a nonnegative integer and let ψ be a positive continuous function on $(0, \infty)$ satisfying $\lim_{x \to \infty} \psi(x)/x^m = \infty$. Then there exists an $R_m > 0$ such that*

$$\sup_{0 < x < \infty} \frac{|p(x)|}{\psi(x)} = \sup_{0 < x \leqslant R_m} \frac{|p(x)|}{\psi(x)} \tag{28}$$

for all polynomials p of degree at most m. If also $\lim_{x \to 0^+} \psi(x) = \infty$, then there exists a $\delta_m > 0$ such that

$$\sup_{0 < x < \infty} \frac{|p(x)|}{\psi(x)} = \sup_{\delta_m \leqslant x \leqslant R_m} \frac{|p(x)|}{\psi(x)} \tag{29}$$

for all polynomials p of degree at most m.

Proof. Let C be the maximum of ψ for $1 \leqslant x \leqslant 3$. Given a polynomial p of degree at most m, let M be the maximum of $|p(x)|/\psi(x)$ for $1 \leqslant x \leqslant 3$. Define $q(t) = p(2+t)$. Then $|q(t)| \leqslant MC$ whenever $-1 \leqslant t \leqslant 1$. By a property of the Chebyshev polynomials [9, p. 52], if $|t| \geqslant 1$ then

$$|q(t)| \leqslant MC\,|T_m(t)| \leqslant MC2^m\,|t|^m. \tag{30}$$

Let $x \geqslant 3$ or $0 < x \leqslant 1$. Applying (30) with $t = x-2$, we obtain

$$\frac{|p(x)|}{\psi(x)} \leqslant MC2^m\,\frac{|x-2|^m}{\psi(x)}. \tag{31}$$

By hypothesis, there exists an $R_m \geqslant 3$ such that $\psi(x)/|x-2|^m > 2^{m+1}C$ for all $x \geqslant R_m$. Hence by (31), we have $|p(x)|/\psi(x) \leqslant M/2$ for all $x \geqslant R_m$. Thus (28) follows.

If the additional hypothesis on ψ holds, then there exists a $\delta_m > 0$ such that $\psi(x)/|x-2|^m > 2^{m+1}C$ whenever $0 < x \leqslant \delta_m$. Hence by (31), we have $|p(x)|/\psi(x) \leqslant M/2$ whenever $0 < x < \delta_m$. Thus (29) follows. ∎

Proof of Theorem 5. To prove part (1) of the theorem, we first consider the case where m is odd with $m>1$ and write $m=2n-1$. (The case $m=1$ is an easy consequence of (i).) Define

$$f_j(u)=\frac{u^j}{\psi(u)}, \qquad j=0, ..., n-1,$$

where $\psi(u)=\phi(\sqrt{u})^m$, and note that $\lim_{u\to\infty}\psi(u)/u^{n-1}=\infty$. Since $f_0, ..., f_{n-2}$ is a Haar system on the interval $I=[0, R_{n-1}]$, there exist numbers $c_0, ..., c_{n-2}$ such that $f\equiv\sum_0^{n-2} c_j f_j$ is the best approximation in the uniform norm on I to f_{n-1} among all such linear combinations. Let $E_m=\|f_{n-1}-f\|_I$. Clearly

$$(f_{n-1}-f)(u)=\frac{\hat{P}(u)}{\psi(u)},$$

where $\hat{P}(u)=u^{n-1}-\sum_{j=0}^{n-2} c_j u^j$. By the Chebyshev alternation theorem [2, p. 74], there exists a decreasing sequence of numbers $u_1, ..., u_n$ in I such that

$$\hat{P}(u_j)=(-1)^{j-1}\,\varepsilon E_m\psi(u_j), \qquad j=1, ..., n, \tag{32}$$

where $\varepsilon=\pm 1$. Comparison of the coefficients of u^{n-1} in the Lagrange interpolation formula for $\hat{P}$ shows that $\varepsilon=1$. Define $P(x)=\hat{P}(x^2)/E_m$. Then P is an even polynomial of degree $m-1$ and $|P(x)|\leqslant\phi(x)^m$ for all real x by (28) of Lemma 9. (Clearly, $A_{m-1}=1/E_m$.)

Define $x_j=\sqrt{u_j}$ for $1\leqslant j\leqslant n$ and $x_j=-x_{m-j+1}$ for $n<j\leqslant m$. Then $P(x_j)=(-1)^{j-1}\,\phi(x_j)^m$ for $1\leqslant j\leqslant m$ since P is even, so P is given by (2). Hence, $x_1, ..., x_m$ is an optimal sequence of ϕ-oscillation points. To show that these points are symmetric with respect to the origin, it suffices to show that $u_n=0$. Let U be the decreasing sequence of numbers $u_1, ..., u_n$. It follows from (12) with $X=U$ and $p=\hat{P}$ that the sum

$$\sum_{j=1}^{n}\frac{\psi(u_j)}{|\omega_j(u_j, U)|} \tag{33}$$

assumes its minimum over all distinct $u_1, ..., u_n$ in I when $u_1, ..., u_n$ are as in (32). If $u_n\neq 0$, then by (i) the above sum is smaller when $u_n=0$, a contradiction.

The proof of part (1) of the theorem for the case where m is even with $m>2$ is similar to the case proved above with $m=2n$, $\psi(u)=\phi(\sqrt{u})^m/\sqrt{u}$, and $I=[\delta_{n-1}, R_{n-1}]$. The optimal sequence of ϕ-oscillation points is defined by the same equations and zero is not one of these points since none of the alternation points is zero. Hence the polynomial P defined by

$P(x) = x\hat{P}(x^2)/E_m$ is an odd polynomial of degree $m-1$ satisfying (2) and $|P(x)| \leqslant \phi(x)^m$ for all real x. (As before, $A_{m-1} = 1/E_m$.)

To treat the case $m=2$, take u_1 to be the point where f_0 assumes its maximum absolute value and proceed as above.

To prove part (2) of Theorem 5, suppose we are given an optimal sequence of ϕ-oscillation points $x_1, ..., x_m$ that are symmetric with respect to the origin, and let p and P satisfy the hypotheses of the theorem. We first consider the case where m is odd. Write $m = 2n-1$ and define $p_2(x) = [p(x) + p(-x)]/2$. Then there exists a polynomial r of degree at most $n-1$ with $p_2(x) = r(x^2)$ for all real x. Put $\psi(u) = \phi(\sqrt{u})^m$ and define $u_j = x_j^2$ for $1 \leqslant j \leqslant n$. By hypothesis,

$$|r(u_j)| \leqslant \psi(u_j), \qquad j = 1, ..., n, \tag{34}$$

so by an elementary result of Rogosinski [10, Theorem I],

$$|r^{(l)}(0)| \leqslant |R^{(l)}(0)|, \qquad l = 0, ..., n-1, \tag{35}$$

where R is the polynomial of degree at most $n-1$ satisfying

$$R(u_j) = (-1)^{j-1}\, \psi(u_j), \qquad j = 1, ..., n. \tag{36}$$

Clearly, $P(x) = R(x^2)$ since both polynomials satisfy (5). Moreover, by [10, Theorem I], if $l > 0$ and equality holds in (35) then $r = \pm R$. Since the coefficients of r (resp., R) are the coefficients of the even powers of p (resp., P), the required inequality (15) follows from (35). Moreover, if equality holds in (15) for an even $k > 0$, then equality holds in (35) for $l = k/2$; therefore, $r = \pm R$ so $p_2 = \pm P$.

If m is even, write $m = 2n$ and define $p_1(x) = [p(x) - p(-x)]/2$. Then there exists a polynomial r of degree no greater than $n-1$ with $p_1(x) = x\, r(x^2)$ for all real x. An argument analogous to the preceding one with $\psi(u) = \phi(\sqrt{u})^m/\sqrt{u}$ and $P(x) = xR(x^2)$ establishes Theorem 5. ∎

6. SAME PARITY PROOF

LEMMA 10. *Let ψ be a positive continuous function on $[0, 1]$. There exists a number r_m with $0 < r_m < 1$ such that*

$$\max_{0 \leqslant x \leqslant 1} \frac{|p(x)|}{\psi(x)} = \max_{0 \leqslant x \leqslant r_m} \frac{|p(x)|}{\psi(x)} \tag{37}$$

for all polynomials p of degree at most m with $p(1) = 0$.

Proof. Define

$$p_1(t) = (t+1)^m\, p\left(\frac{t}{t+1}\right), \qquad \psi_1(t) = (t+1)^m\, \psi\left(\frac{t}{t+1}\right).$$

It follows from the Taylor expansion of p about $t=1$ that p_1 is a polynomial of degree at most $m-1$. Clearly, $\lim_{t\to\infty} \psi_1(t)/t^{m-1} = \infty$. Since the transformation $x = t/(t+1)$ maps $(0, \infty)$ onto $(0, 1)$ and since $p_1(t)/\psi_1(t) = p(x)/\psi(x)$ for $0 < x < 1$, it follows from (28) of Lemma 9 that

$$\sup_{0<x<1} \frac{|p(x)|}{\psi(x)} = \sup_{0<x\leqslant r_m} \frac{|p(x)|}{\psi(x)},$$

where $r_m = R_{m-1}/(1+R_{m-1})$. This implies (37). ∎

Proof of Theorem 6. To prove part (1) of the theorem, we first consider the case where m is odd and write $m = 2n-1$. Define

$$\psi(x) = \begin{cases} |x|\, \phi\left(\dfrac{1}{x}\right) & \text{if } \quad x \neq 0 \\ A & \text{otherwise.} \end{cases}$$

Then ψ is a positive continuous function on $(-\infty, \infty)$ which satisfies $\psi(-x) = \psi(x)$ and $\psi(x) \geqslant \psi(0)$ for all real x, and

$$\lim_{x\to\infty} \frac{\psi(x)}{x} = \phi(0).$$

Hence by part (1) of Theorem 5, there exists an optimal sequence X of ψ-oscillation points $x_1, ..., x_m$ which are symmetric with respect to the origin. Let $w_j = 1/x_j$ for $1 \leqslant j \leqslant m$ with $j \neq n$ (since $x_n = 0$) and define

$$Q_1(x) = x^m P\left(\frac{1}{x}\right), \qquad x \neq 0,$$

where P is the interpolating polynomial corresponding to X. Since P is even and of degree at most $m-1$, Q_1 is an odd polynomial of degree at most m. Moreover,

$$|Q_1(x)| \leqslant |x|^m\, \psi\left(\frac{1}{x}\right)^m = \phi(x)^m, \qquad -\infty < x < \infty,$$

and

$$Q_1(w_j)=\frac{P(x_j)}{x_j^m}=(-1)^{j-1}\frac{\psi(x_j)^m}{x_j^m}=(-1)^{j-1}\,(\text{sign }w_j)\,\phi(w_j)^m$$

for $1\leqslant j\leqslant m$ with $j\neq n$. Note also that $P(0)=(-1)^{n-1}\,\psi(0)^m$ by (5).

Let Y be the sequence of numbers $y_1, ..., y_{m-1}$ obtained by arranging the w_j's into decreasing order. Clearly these numbers are symmetric with respect to the origin. Moreover,

$$Q_1(y_j)=(-1)^{n+j-1}\,\phi(y_j)^m, \qquad j=1, ..., m-1,$$

and

$$\lim_{x\to\infty}\frac{Q_1(x)}{x^m}=P(0)=(-1)^{n-1}\,A^m.$$

Define $Q=(-1)^{n-1}\,Q_1$. Then Q satisfies (6) and (7) so Q is given by (4). Thus Y is an optimal sequence of (ϕ, ∞)-oscillation points.

Next we prove part (1) of the theorem when m is even. Write $m=2n$ and let $l=[n/2]$. We may assume that $n\geqslant 2$ since the case $m=2$ is easy to verify directly. Define

$$f_j(u)=\frac{u^j+(-1)^n\,u^{n-j}}{\psi(u)}, \qquad j=0, ..., l,$$

where $\psi(u)=\phi(\sqrt{u})^m$, and let J be the set of all integers j satisfying $0\leqslant j\leqslant l$ and $j\neq 1$. We begin by showing that if n is even then $\{f_j : j\in J\}$ is a Haar system on $[0, 1]$. Since this is obvious when $n=2$, suppose that $n\geqslant 4$. Let $f=\sum_{j\in J} c_j\, f_j$, where $c_j\neq 0$ for at least one j in J. Then $f(u)$ is a nontrivial linear combination of at most $n-1$ distinct positive powers of u including $u^0=1$. Hence f has at most $n-2$ roots in $[0, \infty)$ since these powers of u are a Haar system on $[0, \infty)$. If 0 is a root of f, then $c_0=0$ so the same argument shows that f has at most $n-4$ roots in $(0, \infty)$.

Suppose f has at least l roots in $[0, 1]$. Note that the reciprocal of a nonzero root of f is a root of f since $f(u^{-1})=(-1)^n\, f(u)$ for all real $u\neq 0$. If 0 is not a root of f, then f has at least $l-1$ roots in $(1, \infty)$ and hence at least $l+(l-1)=n-1$ roots in $(0, \infty)$, a contradiction. If 0 is a root of f, then f has at least $l-2$ roots in $(1, \infty)$ and hence at least $(l-1)+(l-2)=n-3$ roots in $(0, \infty)$, a contradiction. Thus $\{f_j : j\in J\}$ is a Haar system on $[0, 1]$. A slight modification of this argument shows that $\{f_j : j\in J\}$ is a Haar system on $[0, 1)$ when n is odd. (There are counterexamples for $[0, 1]$.)

Let $f=\sum_{j\in J} c_j f_j$ and note that

$$(f_1-f)(u)=\frac{\hat{Q}(u)}{\psi(u)},$$

where

$$\hat{Q}(u)=P_1(u)+(-1)^n u^n P_1\left(\frac{1}{u}\right),$$

$$P_1(u)=u-(c_0+c_2u^2+\cdots+c_l u^l).$$

Since $\hat{Q}(1)=0$ when n is odd, by Lemma 10 and the Haar systems established above, there is a best approximation f to f_1 in the uniform norm on $[0,1]$ among all such linear combinations. Let $F_m=\|f_1-f\|_I$, where $I=[0,1]$. Applying the Chebyshev alternation theorem in the same way, we obtain a decreasing sequence of numbers $u_1, ..., u_{l+1}$ in $[0,1]$ such that

$$\hat{Q}(u_j)=(-1)^{j-1}\varepsilon F_m\psi(u_j), \qquad j=1, ..., l+1, \tag{38}$$

where $\varepsilon=\pm 1$. Since $|\hat{Q}(u)|\leqslant F_m\psi(u)$ for all $u\in[0,1]$, it follows from the identities

$$u^n\hat{Q}\left(\frac{1}{u}\right)=(-1)^n\hat{Q}(u), \qquad u^n\psi\left(\frac{1}{u}\right)=\psi(u) \tag{39}$$

that the same inequality holds for all $u\geqslant 0$.

To show that $u_{l+1}=0$, suppose $u_{l+1}\neq 0$. By (38) and (39), we have

$$\hat{Q}\left(\frac{1}{u_j}\right)=(-1)^n\frac{\hat{Q}(u_j)}{u_j^n}=(-1)^{n+j-1}\varepsilon F_m\psi\left(\frac{1}{u_j}\right) \tag{40}$$

for all $j=1, ..., l+1$. Let U be the sequence with $n+1$ terms obtained by arranging the numbers $1/u_j$ and u_j, $j=1, ..., l+1$, in decreasing order and omitting u_1 when n is even. Then an argument analogous to that using (33) shows that $u_{l+1}=0$, the desired contradiction.

Define $Q(x)=(-1)^{n+l}\hat{Q}(x^2)/(\varepsilon F_m)$. Then Q is an even polynomial of degree at most m and $|Q(x)|\leqslant\phi(x)^m$ for all real x. Since $u_{l+1}=0$, we have $Q(0)=(-1)^n\phi(0)^m$ so

$$\lim_{x\to\infty}\frac{Q(x)}{\phi(x)^m}=\lim_{x\to 0^+}\frac{Q(1/x)}{\phi(1/x)^m}=\lim_{x\to 0^+}\frac{(-1)^n Q(x)}{\phi(x)^m}=1.$$

(Clearly $B_{m-2}=-1/F_m$ for $m>4$ and $B_{m-2}=-2/F_m$ if $m=4$.)

Let $v_1, ..., v_n$ be the sequence obtained by arranging the numbers $1/u_j$, $j=1, ..., l$, and u_j, $j=1, ..., l+1$, in decreasing order and omitting u_1 if n is even. Define a decreasing sequence Y by $y_j=\sqrt{v_j}$ for $1\leqslant j\leqslant n$ and $y_j=-y_{m-j}$ for $n<j\leqslant m-1$. Then the points of Y are symmetric with respect to the origin (since $y_n=0$) and (6) holds by (38) and (40).

Finally, we sketch the proof of the inequalities (18). It follows from the hypothesis (17) that for each $r>1$ there is a number $N>y_1$ such that $|p(t)|\leqslant r\phi(t)^m$ whenever $t\geqslant N$. Put $p_r=p/r$. Define $y_0=t$ and $y_m=-t$, and let P_t be defined as in (8). Then $|p_r(y_j)|\leqslant\phi(y_j)^m$ for all $j=0, ..., m$ by (16). By an argument analogous to the proof of part 2 of Theorem 6 (or, better, see [10, Theorem III]), it follows that

$$|p_r^{(k)}(0)|\leqslant|P_t^{(k)}(0)| \tag{41}$$

for all k with the same parity as m and $0\leqslant k\leqslant m$. Taking limits in (41) as $t\to\infty$ and then as $r\to1^+$, we obtain the desired inequalities (18).

If equality holds in (18), then equality almost holds in (41) for all r close to 1 and all large t. A modification of the argument given in [10] shows that the sides of the asserted equality differ at most by a number which approaches 0 as $r\to1^+$ and $t\to\infty$. ∎

Proof of Proposition 8. Note that the interpolating polynomials obtained in the proofs of part (1) of Theorems 5 and 6 satisfy the identities (19), (20), (22), and (23). Thus we need to prove only (21). Let m be even and let $x_1, ..., x_m$ be an optimal sequence of ϕ-oscillation points which are symmetric with respect to the origin. Define $R(x)=x^mP(1/x)$. Clearly $P(0)=0$ by (19), so R is a polynomial of degree at most $m-1$. Also, none of the points x_j can be zero, and hence we may define $w_j=1/x_j$ for $j=1, ..., m$. By (iv), we have $|R(x)|\leqslant\phi(x)^m$ for all real x and

$$R(w_j)=\frac{P(x_j)}{x_j^m}=(-1)^{j-1}\,\frac{\phi(x_j)^m}{x_j^m}=(-1)^{j-1}\,\phi(w_j)^m$$

for $j=1, ..., m$. Let $v_1, ..., v_m$ be the sequence obtained by arranging the w_j's into decreasing order. Then it is easy to verify that $R(v_j)=(-1)^{n-j}\,\phi(v_j)^m$ for $j=1, ..., m$, where $n=m/2$. Hence $v_1, ..., v_m$ is an optimal sequence of ϕ-oscillation points and $(-1)^{n-1}R$ is the corresponding interpolating polynomial so $(-1)^{n-1}R=P$ by Proposition 4. ∎

There are alternate proofs of (19) and (20) which show that these hold for any optimal sequences of ϕ- and (ϕ,∞)-oscillation points, respectively. Indeed, one can deduce (19) directly from Proposition 4 by observing that $(-1)^{m-1}P(-x)$ is the corresponding interpolating polynomial for the optimal sequence W of ϕ-oscillation points defined by $w_j=-x_{m-j+1}$ for $j=1, ..., m$. A similar argument establishes (20).

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